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Collapse models: analysis of the free particle dynamics

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Abstract
We study a model of spontaneous wavefunction collapse for a free quantum particle. We analyse in detail the time evolution of the single-Gaussian solution and the double-Gaussian solution, showing how the reduction mechanism induces the localization of the wavefunction in space; we also study the asymptotic behaviour of the general solution. With an appropriate choice for the parameter λ which sets the strength of the collapse mechanism we prove that: (i) the effects of the reducing terms on the dynamics of microscopic systems are negligible, the physical predictions of the model being very close to those of standard quantum mechanics; (ii) at the macroscopic scale the model reproduces classical mechanics: the wavefunction of the centre of mass of a macro-object behaves, with high accuracy, like a point moving in space according to Newton's laws.

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1. Introduction

Models of spontaneous wavefunction collapse [1–15] have reached significant results in providing a solution to the measurement problem of quantum mechanics. This goal is achieved by modifying the Schrödinger equation, adding appropriate nonlinear stochastic terms[1]: such terms do not modify appreciably the standard quantum dynamics of microscopic systems; at the same time they rapidly reduce the superposition of two or more macroscopically different states of a macro-object into one of them; in particular they guarantee that measurements made on microscopic systems always have definite outcomes, and with the

[1] If one aims at reproducing the process of wavefunction collapse in measurement-like situation, the terms which have to be added to the Schrödinger equation must be nonlinear and stochastic, since these two are the characteristic features of the quantum collapse process.

correct quantum probabilities. In this way, collapse models describe—within one single dynamical framework—both the quantum properties of microscopic systems and the classical properties of macroscopic objects, providing a unified description of micro- and macro-phenomena.

In this paper, we investigate the physical properties of a collapse model describing the (one-dimensional) evolution of a free quantum particle subject to spontaneous localizations in space; its dynamics is governed by the following stochastic differential equation in the Hilbert space $L^2(\mathbb{R})$:

$$\mathrm{d}\psi_t(x) = \left[-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\,\mathrm{d}t + \sqrt{\lambda}(q - \langle q\rangle_t)\,\mathrm{d}W_t - \frac{\lambda}{2}(q - \langle q\rangle_t)^2\,\mathrm{d}t \right]\psi_t(x), \tag{1}$$

where q and p are the position and momentum operators, respectively, and $\langle q\rangle_t \equiv \langle\psi_t|q|\psi_t\rangle$ denotes the quantum average of the operator q; m is the mass of the particle while λ is a positive constant which sets the strength of the collapse mechanism. The stochastic dynamics is governed by a standard Wiener process W_t defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t : t \geqslant 0\}$ defined on it.

The value of the collapse constant λ is given by the formula[2]:

$$\lambda = \frac{m}{m_0}\lambda_0, \tag{2}$$

where m_0 is a reference mass which we choose to be equal to that of a nucleon and λ_0 is a fixed constant which we take equal to

$$\lambda_0 \simeq 10^{-2}\mathrm{m}^{-2}\,\mathrm{s}^{-1}; \tag{3}$$

this value corresponds to the product of the two parameters Λ_{GRW} and α_{GRW} of the GRW collapse model [1], where $\Lambda_{\mathrm{GRW}} \simeq 10^{-16}\,\mathrm{s}^{-1}$ is the localization rate for a nucleon and $1/\sqrt{\alpha_{\mathrm{GRW}}} \simeq 10^{-7}$ m is the width of the Gaussian wavefunction inducing the localizations.

Equation (1) has already been subject to investigation: in [10], a theorem proves the existence and uniqueness of strong solutions[3]; in [5, 8, 10, 20], some properties of the solutions have been analysed: in particular, it has been shown that Gaussian wavefunctions are solutions of equation (1), that their spread reaches an asymptotic finite value (we speak in this case of a ‘stationary’ solution), and that the general solution reaches asymptotically a stationary Gaussian solution. Finally in [6], equation (1) has been first proposed as a universal model of wavefunction collapse.

The aim of our work is to provide a detailed analysis of the physical properties of the solutions of equation (1). After some mathematical preliminaries (section 2) we will study the time evolution of the two most interesting types of wavefunctions: the single-Gaussian (section 3) and the double-Gaussian wavefunctions (section 4); in section 5 we will discuss the asymptotic behaviour of the general solution.

We will next study the effects of the stochastic dynamics on microscopic systems (section 6) and on macroscopic objects (section 7); in the first case, we will see that the prediction of the model are very close to those of standard quantum mechanics, while in the second case we will show that the wavefunction of a macro-object is well localized in space and behaves like a point moving in space according to the classical laws of motion. We end up with some concluding remarks (section 8).

[2] Differently from [6], we assume that λ is proportional to the mass of the particle.

[3] Existence and uniqueness theorems for a wide class of stochastic differential equations whose coefficients are *bounded* operators are studied in [16]; the case of *unbounded* operators is covered in [17]. See [18, 19] for an introduction to stochastic differential equation in infinite-dimensional spaces.

2. Linear versus nonlinear equation

The easiest way to find solutions of a nonlinear equation is—when feasible—to linearize it: this is possible for equation (1) and the procedure is well known in the literature [2, 3, 10, 15–17]. Let us consider the following linear stochastic differential equation:

$$\mathrm{d}\phi_t(x) = \left[-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\,\mathrm{d}t + \sqrt{\lambda}q\,\mathrm{d}\xi_t - \frac{\lambda}{2}q^2\,\mathrm{d}t\right]\phi_t(x); \tag{4}$$

ξ_t is a standard Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ is a new probability measure whose connection with $\mathbb{P}$ will be clear in what follows. Contrary to equation (1) the above equation does not preserve the norm of statevectors, so let us define the normalized vectors:

$$\psi_t = \begin{cases} \phi_t/\|\phi_t\| & \text{if} \quad \|\phi_t\| \neq 0, \\ \text{a fixed unit vector} & \text{if} \quad \|\phi_t\| = 0. \end{cases} \tag{5}$$

By using Itô calculus it is not difficult to show that ψ_t defined by (5) is a solution of equation (1), whenever ϕ_t solves equation (4). We now briefly explain the relations between the two probability measures $\mathbb{Q}$ and $\mathbb{P}$ and between the two Wiener processes ξ_t and W_t.

The key property of equation (4) is that $p_t \equiv \|\phi_t\|^2$ is a martingale [10, 17] satisfying the following stochastic differential equation:

$$\mathrm{d}p_t = 2\sqrt{\lambda}\langle q\rangle_t\, p_t\,\mathrm{d}\xi_t, \tag{6}$$

with $\langle q\rangle_t = \langle\psi_t|q|\psi_t\rangle$. As a consequence of the martingale property (and assuming, as we shall always do, that $\|\phi_0\| = 1$), p_t can be used to generate a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ [19]; we choose $\mathbb{Q}$ in such a way that the new measure $\tilde{\mathbb{P}}$ coincides with $\mathbb{P}$.

Given this, Girsanov's theorem [21] provides a simple relation between the Wiener process ξ_t defined on $(\Omega, \mathcal{F}, \mathbb{Q})$ and the Wiener process W_t defined on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$W_t = \xi_t - 2\sqrt{\lambda}\int_0^t \langle q\rangle_s\,\mathrm{d}s. \tag{7}$$

The above results imply that one can find the solution ψ_t of equation (1), given the initial condition ψ_0, by using the following procedure:

1. Find the solution ϕ_t of equation (4), with the initial condition $\phi_0 = \psi_0$.
2. Normalize the solution: $\phi_t \to \psi_t = \phi_t/\|\phi_t\|$.
3. Make the substitution: $\mathrm{d}\xi_t \to \mathrm{d}W_t + 2\sqrt{\lambda}\langle q\rangle_t$.

The advantage of such an approach is that one can exploit the linear character of equation (4) to analyse the properties of the nonlinear equation (1); as we shall see, the difficult part comes when one has to compute $\langle q\rangle_t$ from ϕ_t (step 3): this is where nonlinearity enters in a non-trivial way.

3. Single Gaussian solution

We start our analysis by taking, as a solution, a single-Gaussian wavefunction[4]:

$$\phi_t^{\mathrm{S}}(x) = \exp[-a_t(x - \overline{x}_t)^2 + \mathrm{i}\overline{k}_t x + \gamma_t], \tag{8}$$

[4] See [22] for an analogous discussion within the CSL [2] model of wavefunction collapse.

where a_t and γ_t are supposed to be complex functions of time, while $\overline{x}_t$ and $\overline{k}_t$ are taken to be real[5]. By inserting (8) into equation (4) one finds the following set of stochastic differential equations[6]:

$$\mathrm{d}a_t = \left[\lambda - \frac{2\mathrm{i}\hbar}{m}(a_t)^2\right]\mathrm{d}t \tag{9}$$

$$\mathrm{d}\overline{x}_t = \frac{\hbar}{m}\overline{k}_t\,\mathrm{d}t + \frac{\sqrt{\lambda}}{2a_t^{\mathrm{R}}}[\mathrm{d}\xi_t - 2\sqrt{\lambda}\overline{x}_t\,\mathrm{d}t] \tag{10}$$

$$\mathrm{d}\overline{k}_t = -\sqrt{\lambda}\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}[\mathrm{d}\xi_t - 2\sqrt{\lambda}\overline{x}_t\,\mathrm{d}t] \tag{11}$$

$$\mathrm{d}\gamma_t^{\mathrm{R}} = \left[\lambda\overline{x}_t^2 + \frac{\hbar}{m}a_t^{\mathrm{I}}\right]\mathrm{d}t + \sqrt{\lambda}\overline{x}_t[\mathrm{d}\xi_t - 2\sqrt{\lambda}\overline{x}_t\,\mathrm{d}t] \tag{12}$$

$$\mathrm{d}\gamma_t^{\mathrm{I}} = \left[-\frac{\hbar}{m}a_t^{\mathrm{R}} - \frac{\hbar}{2m}\overline{k}_t^2\right]\mathrm{d}t + \sqrt{\lambda}\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\overline{x}_t[\mathrm{d}\xi_t - 2\sqrt{\lambda}\overline{x}_t\,\mathrm{d}t]. \tag{13}$$

For a single-Gaussian wavefunction, the two equations for γ_t can be omitted since the real part of γ_t is absorbed into the normalization factor, while the imaginary part gives an irrelevant global phase.

The normalization procedure is trivial, and also the Girsanov transformation (7) is easy since, for a Gaussian wavefunction like (8), one simply has $\langle q\rangle_t = \overline{x}_t$. We then have the following set of stochastic differential equations for the relevant parameters:

$$\mathrm{d}a_t = \left[\lambda - \frac{2\mathrm{i}\hbar}{m}(a_t)^2\right]\mathrm{d}t, \tag{14}$$

$$\mathrm{d}\overline{x}_t = \frac{\hbar}{m}\overline{k}_t\,\mathrm{d}t + \frac{\sqrt{\lambda}}{2a_t^{\mathrm{R}}}\,\mathrm{d}W_t, \tag{15}$$

$$\mathrm{d}\overline{k}_t = -\sqrt{\lambda}\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\,\mathrm{d}W_t; \tag{16}$$

throughout this section we will discuss the physical implications of these equations.

3.1. The equation for a_t

Equation (14) for a_t can easily be solved [8, 10]:

$$a_t = c\tanh\left[bt + k\right], \tag{17}$$

with

$$c = (1-\mathrm{i})\frac{1}{2}\sqrt{\frac{m\lambda}{\hbar}}, \qquad b = (1+\mathrm{i})\sqrt{\frac{\hbar\lambda}{m}}, \qquad k = \tanh^{-1}\left[\frac{a_0}{c}\right]. \tag{18}$$

After some algebra one obtains the following analytical expressions for the real and the imaginary parts of a_t:

$$a_t^{\mathrm{R}} = \frac{\lambda}{\omega}\frac{\sinh(\omega t + \varphi_1) + \sin(\omega t + \varphi_2)}{\cosh(\omega t + \varphi_1) + \cos(\omega t + \varphi_2)}, \tag{19}$$

[5] For simplicity, we will assume in the following that the initial values of these parameters do not depend on $\omega \in \Omega$.

[6] The superscripts 'R' and 'I' denote, respectively, the real and imaginary parts of the corresponding quantities.

$$a_t^{\mathrm{I}} = -\frac{\lambda}{\omega}\frac{\sinh(\omega t + \varphi_1) - \sin(\omega t + \varphi_2)}{\cosh(\omega t + \varphi_1) + \cos(\omega t + \varphi_2)}, \tag{20}$$

where we have defined the frequency:

$$\omega = 2\sqrt{\frac{\hbar\lambda_0}{m_0}} \simeq 10^{-5}\,\mathrm{s}^{-1}, \tag{21}$$

which does not depend on the mass of the particle. The two parameters φ_1 and φ_2 are functions of the initial condition: $\varphi_1 = 2k^{\mathrm{R}}$, $\varphi_2 = 2k^{\mathrm{I}}$.

An important property of a_t^{R} is positivity:

$$a_0^{\mathrm{R}} > 0 \quad \longrightarrow \quad a_t^{\mathrm{R}} > 0 \qquad \forall t > 0, \tag{22}$$

which guarantees that an initially Gaussian wavefunction does not diverge at any later time. To prove this, we first note that the denominator of (19) cannot be negative; if it is equal to zero then also the numerator is zero and the discontinuity can be removed by using expression (17) for a_t, which—according to the values (18) for b and k—is analytic for any t. It is then sufficient to show that the numerator remains positive throughout time. Let us consider the function $f(t) = \sinh(\omega t + \varphi_1) + \sin(\omega t + \varphi_2)$; we have that $f(0) > 0$ and $f'(t) \geqslant 0$ for any t, which implies that $f(t) > 0$ for any t, as desired. Note that positivity of a_t^{R} matches with the fact that equation (1) preserves the norm of statevectors.

3.2. *The spread in position and momentum*

The time evolution of the spread in position and momentum of the Gaussian wavefunction (8),

$$\begin{aligned} \sigma_q(t) &= \sqrt{\langle q^2\rangle - \langle q\rangle^2} = \frac{1}{2}\sqrt{\frac{1}{a_t^{\mathrm{R}}}}, \\ \sigma_p(t) &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \hbar\sqrt{\frac{\left(a_t^{\mathrm{R}}\right)^2 + \left(a_t^{\mathrm{I}}\right)^2}{a_t^{\mathrm{R}}}}, \end{aligned} \tag{23}$$

is given by the following analytical expressions:

$$\sigma_q(t) = \sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{\cosh(\omega t + \varphi_1) + \cos(\omega t + \varphi_2)}{\sinh(\omega t + \varphi_1) + \sin(\omega t + \varphi_2)}}, \tag{24}$$

$$\sigma_p(t) = \sqrt{\frac{\hbar m\omega}{2}}\sqrt{\frac{\cosh(\omega t + \varphi_1) - \cos(\omega t + \varphi_2)}{\sinh(\omega t + \varphi_1) + \sin(\omega t + \varphi_2)}}. \tag{25}$$

Figure 1 shows the different time dependence of the spread in position, as given by the Schrödinger equation and by the stochastic equation: as we see, at the beginning the two evolutions almost coincide; as time increases, while in the standard quantum case the spread goes to infinity, the spread according to our stochastic equation reaches the asymptotic value:

$$\sigma_q(\infty) = \sqrt{\frac{\hbar}{m\omega}} \simeq \left(10^{-15}\sqrt{\frac{\mathrm{kg}}{m}}\right)\mathrm{m}, \tag{26}$$

which of course depends on the mass m of the particle. This behaviour can be understood as follows: the reduction terms (which tend to localize the wavefunction) and the standard quantum Hamiltonian (which tends to spread out the wavefunction) compete against each other until an equilibrium—the stationary solution—is reached, which depends on the values of the parameters of the model.

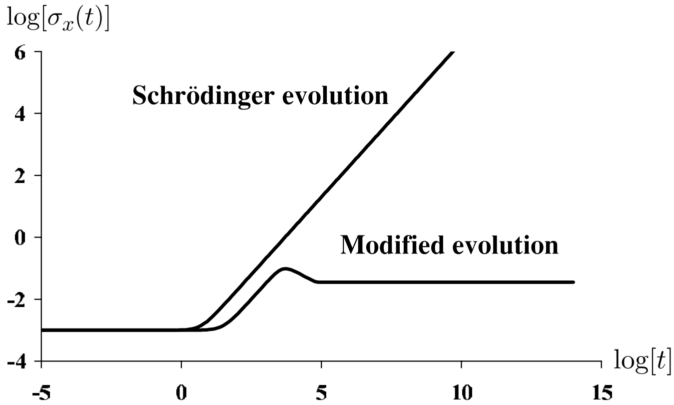

Figure 1. The picture shows the time evolution (t measured in seconds) of the spread in position $\sigma_q(t)$ (measured in metres) of a Gaussian wavefunction, as given by the Schrödinger equation and by the stochastic equation. The simulation has been made for a nucleon ($m = m_0$). The initial spread has been taken equal to 10^{-3} m.

Also the spread in momentum changes in time, reaching the asymptotic value:

$$\sigma_p(\infty) = \sqrt{\frac{\hbar m\omega}{2}} \simeq \left(10^{-19}\sqrt{\frac{m}{\text{kg}}}\right)\frac{\text{kg m}}{\text{s}}. \tag{27}$$

It is interesting to compare the two asymptotic values for the spread in position and momentum; one has

$$\sigma_q(\infty)\,\sigma_p(\infty) = \frac{\hbar}{\sqrt{2}} \tag{28}$$

which corresponds to almost the minimum allowed by Heisenberg's uncertainty relations [1, 20]: the collapse model then induces almost the best possible localization of the wavefunction—both in position and momentum. In accordance with [5] any Gaussian wavefunction having these asymptotic values for σ_q and σ_p will be called a 'stationary solution'[7] of equation (1).

Note the interesting fact that the evolution of the spread in position and momentum is *deterministic* and depends on the noise only indirectly, through the constant λ.

3.3. The mean in position and momentum

The quantities $\langle q\rangle_t = \overline{x}_t$ and $\langle p\rangle_t = \hbar\overline{k}_t$, corresponding to the peak of the Gaussian wavefunction in the position and momentum spaces, respectively, satisfy the following stochastic differential equations:

$$\mathrm{d}\langle q\rangle_t = \frac{1}{m}\langle p\rangle_t\,\mathrm{d}t + \sqrt{\lambda}\frac{1}{2a_t^{\mathrm{R}}}\,\mathrm{d}W_t, \tag{29}$$

$$\mathrm{d}\langle p\rangle_t = -\sqrt{\lambda}\hbar\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\,\mathrm{d}W_t. \tag{30}$$

[7] Strictly speaking a 'stationary' solution is not stationary at all since both the mean in position and the mean in momentum may change in time—as it happens in our case; the term 'stationary' refers only to the spread of the wavefunction.

Their average values obey the *classical* equations[8] for a free particle of mass m:

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\langle q\rangle_t\right] = \mathbb{E}\left[\langle p\rangle_t\right], \tag{31}$$

$$\mathbb{E}\left[\langle p\rangle_t\right] = \langle p\rangle_0, \tag{32}$$

while the coefficients of covariance matrix

$$C(t) = \mathbb{E}\left[\begin{bmatrix}\langle q\rangle_t - \mathbb{E}[\langle q\rangle_t]\\ \langle p\rangle_t - \mathbb{E}[\langle p\rangle_t]\end{bmatrix}\cdot\begin{bmatrix}\langle q\rangle_t - \mathbb{E}[\langle q\rangle_t]\\ \langle p\rangle_t - \mathbb{E}[\langle p\rangle_t]\end{bmatrix}^{\top}\right] \equiv \begin{bmatrix}C_{q^2}(t) & C_{qp}(t)\\ C_{pq}(t) & C_{p^2}(t)\end{bmatrix}$$

evolve as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{q^2}(t) = \frac{2}{m}C_{qp}(t) + \frac{\lambda}{4}\frac{1}{\left(a_t^{\mathrm{R}}\right)^2}, \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{qp}(t) = \frac{1}{m}C_{p^2}(t) - \frac{\lambda\hbar}{2}\frac{a_t^{\mathrm{I}}}{\left(a_t^{\mathrm{R}}\right)^2}, \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{p^2}(t) = \lambda\hbar^2\left(\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\right)^2. \tag{35}$$

Particularly interesting is the third equation, which implies that the wavefunction picks larger and larger components in momentum, as time increases; as a consequence the energy of the system increases in time, as can be seen by writing down the stochastic differential for $\langle H\rangle_t \equiv \langle p^2\rangle_t/2m$:

$$\mathrm{d}\langle H\rangle_t = \frac{\lambda\hbar^2}{2m}\,\mathrm{d}t - \sqrt{\lambda}\frac{\hbar}{m}\frac{a_t^{\mathrm{I}}}{a_t^{\mathrm{R}}}\langle p\rangle_t\,\mathrm{d}W_t, \tag{36}$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}[\langle H\rangle_t] = \frac{\lambda\hbar^2}{2m} = \frac{\lambda_0\hbar^2}{2m_0} \simeq 10^{-43}\,\mathrm{J\,s^{-1}}. \tag{37}$$

This energy non-conservation is a typical feature of space-collapse models, but with our choice for the parameter λ the increase is so weak that it cannot be detected with present-day technology (see [1]).

4. Double Gaussian solution

We now study the time evolution of the superposition of two Gaussian wavefunctions; such an analysis is interesting since it allows to understand in a quite simple and clear way how the reduction mechanism works, i.e. how the superposition of two different position states is reduced into one of them. To this purpose, let us consider the following wavefunction:

$$\begin{aligned}\phi_t^{\mathrm{D}}(x) = \phi_{1t}(x) + \phi_{2t}(x) &= \exp[-a_{1t}(x-\overline{x}_{1t})^2 + \mathrm{i}\overline{k}_{1t}x + \gamma_{1t}]\\ &\quad + \exp[-a_{2t}(x-\overline{x}_{2t})^2 + \mathrm{i}\overline{k}_{2t}x + \gamma_{2t}];\end{aligned} \tag{38}$$

we follow the strategy outlined in section 2, by first finding the solution of the linear equation.

Because of linearity $\phi_t^{\mathrm{D}}(x)$ is automatically a solution of equation (4) provided that its parameters satisfy equation (9)–(13). The difficult part of the analysis is related to the change of measure: the reason is that in the double-Gaussian case the quantum average $\langle q\rangle_t$ is not simply equal to $\overline{x}_{1t}$ or $\overline{x}_{2t}$, as it is for a single-Gaussian wavefunction, but is a non-trivial

[8] These equations can be considered as the stochastic extension of Ehrenfest's theorem.

function[9] of all the parameters defining $\phi_t^{\mathrm{D}}(x)$; in spite of this difficulty the most interesting properties of the dynamical evolution of $\phi_t^{\mathrm{D}}(x)$ can be analysed in a rigorous way.

We first observe that the two equations for a_{1t} and a_{2t} are deterministic and thus insensitive of the change of measure; accordingly, the spread (both in position and in momentum) of the two Gaussian functions defining $\phi_t^{\mathrm{D}}(x)$ evolve *independently* of each other, and maintain all the properties discussed in the previous section. For simplicity, we assume that $a_{1t} = a_{2t}$ at $t = 0$ so that these two parameters will remain equal at any subsequent time.

4.1. The asymptotic behaviour

Let us consider the differences $X_t = \overline{x}_{2t} - \overline{x}_{1t}$ and $K_t = \overline{k}_{2t} - \overline{k}_{1t}$ between the peaks of the two Gaussian functions in the position and in the momentum spaces; they satisfy the following set of equations:

$$\mathrm{d}\begin{bmatrix} X_t \\ K_t \end{bmatrix} = \begin{bmatrix} -A_1(t) & \hbar/m \\ -A_2(t) & 0 \end{bmatrix} \cdot \begin{bmatrix} X_t \\ K_t \end{bmatrix} \mathrm{d}t, \tag{39}$$

with

$$A_1(t) = \omega \frac{\cosh(\omega t + \varphi_1) + \cos(\omega t + \varphi_2)}{\sinh(\omega t + \varphi_1) + \sin(\omega t + \varphi_2)}, \tag{40}$$

$$A_2(t) = 2\lambda \frac{\sinh(\omega t + \varphi_1) - \sin(\omega t + \varphi_2)}{\sinh(\omega t + \varphi_1) + \sin(\omega t + \varphi_2)}. \tag{41}$$

We see—this is the reason why we have taken into account the differences X_t and K_t—that the above system of equations is *deterministic*, so it does not depend on the change of measure.

The coefficients of the 2×2 matrix $A(t)$ defining the linear system (39) are analytic in the variable t, and the Liapunov's type numbers [24] of the system are the same as those of the linear system obtained by replacing $A(t)$ with $A(\infty)$, where

$$A(\infty) \equiv \lim_{t \to +\infty} A(t) = \begin{bmatrix} -\omega & \hbar/m \\ -2\lambda & 0 \end{bmatrix}. \tag{42}$$

The eigenvalues of the matrix $A(\infty)$ are

$$\mu_{1,2} = -\tfrac{1}{2}\,(1 \pm \mathrm{i})\,\omega, \tag{43}$$

from which it follows that the linear system (39) has only one Lyapunov's type number: $-\omega/2$. This implies that for any non-trivial (vector) solution $\mathbf{Z}(t)$ of (39), one has

$$\lim_{t \to +\infty} \frac{\ln|\mathbf{Z}(t)|}{t} = -\frac{\omega}{2} \quad \Rightarrow \quad \lim_{t \to +\infty} \mathbf{Z}(t) = \mathbf{0}. \tag{44}$$

We arrive at the following result: asymptotically the difference $\overline{x}_{2t} - \overline{x}_{1t}$ between the peaks of the two Gaussian wavefunctions in the position space goes to zero; also the difference $\overline{k}_{2t} - \overline{k}_{1t}$ between the peaks of the two Gaussian wavefunctions in the momentum space vanishes. In other words, the two Gaussian wavefunctions converge towards each other, and asymptotically they become one single-Gaussian wavefunction which, from the analysis of the previous subsection, is a 'stationary' solution of the stochastic equation (1).

Anyway, this behaviour in general *cannot* be responsible for the collapse of a macroscopic superposition; as a matter of fact, let us consider the following situation which, for later convenience, we call situation $\mathcal{A}$:

[9] This is the reason why the dynamics of these parameters changes in a radical way with respect to the single-Gaussian case.

(i) The wavefunction is in a superposition of the form (38) and, at time $t = 0$ (consequently, also for any later t), a_{1t} and a_{2t} are equal to their asymptotic value.

(ii) The distance $K_t \equiv \overline{k}_{2t} - \overline{k}_{1t}$ is zero at time $t = 0$.

Under these assumptions, the linear system (39) can easily be solved and one gets

$$X_t = X_0\, \mathrm{e}^{-\omega t/2}\left[\cos\frac{\omega t}{2} - \sin\frac{\omega t}{2}\right]. \tag{45}$$

We see that the time evolution of X_t is independent of the mass of the particle: this result implies that, when the spread of the two Gaussian wavefunctions is equal to its asymptotic value[10], their distance decreases with a rate $\omega/2 \sim 10^{-5}\ \mathrm{s}^{-1}$ which is too slow to justify a possible collapse, in particular at the macro-level.

4.2. The collapse

Now we show that the collapse of the wavefunction occurs because, during the evolution, one of the two Gaussian wavefunctions is suppressed with respect to the other one[11]; the quantity which measure this damping is the difference $\Gamma_t^{\mathrm{R}} = \gamma_{2t}^{\mathrm{R}} - \gamma_{1t}^{\mathrm{R}}$, which satisfies the stochastic differential equation:

$$\mathrm{d}\Gamma_t^{\mathrm{R}} = -\lambda X_t[\,\overline{x}_{1t} + \overline{x}_{2t} - 2\langle q\rangle_t]\mathrm{d}t + \sqrt{\lambda}X_t\,\mathrm{d}W_t; \tag{46}$$

if $\Gamma_t^{\mathrm{R}} \to +\infty$, then ϕ_1 is suppressed with respect to ϕ_2 and the superposition practically reduces to ϕ_2; the opposite happens if $\Gamma_t^{\mathrm{R}} \to -\infty$.

To be more precise, we introduce a positive constant b which we assume to be conveniently large (let us say, $b = 10$) and we say that the superposition is *suppressed* when $|\Gamma^{\mathrm{R}}| \geqslant b$; moreover, we say that

$$\phi^{\mathrm{D}} \text{ is reduced to } \phi_1 \text{ when } \Gamma^{\mathrm{R}} \leqslant -b,$$
$$\phi^{\mathrm{D}} \text{ is reduced to } \phi_2 \text{ when } \Gamma^{\mathrm{R}} \geqslant +b.$$

We now study the time evolution of Γ_t^{R}.

By writing $\langle q\rangle_t$ in terms of the coefficients defining $\phi_t^{\mathrm{D}}(x)$:

$$\langle q\rangle_t = \sqrt{\frac{\pi}{2a_t^{\mathrm{R}}}}\left[\overline{x}_{1t}\,\mathrm{e}^{2\gamma_{1t}^{\mathrm{R}}} + \overline{x}_{2t}\,\mathrm{e}^{2\gamma_{2t}^{\mathrm{R}}} + \delta_t\,\mathrm{e}^{\gamma_{1t}^{\mathrm{R}}+\gamma_{2t}^{\mathrm{R}}}\right] \tag{47}$$

it is not difficult to prove that equation (46) becomes

$$\mathrm{d}\Gamma_t^{\mathrm{R}} = \lambda X_t^2 \tanh\Gamma_t^{\mathrm{R}}\,\mathrm{d}t + g_t\left(\Gamma_t^{\mathrm{R}}\right)\mathrm{d}t + \sqrt{\lambda}X_t\,\mathrm{d}W_t, \tag{48}$$

where we have defined the following quantities:

$$\delta_t = h_t\,[(\overline{x}_{1t} + \overline{x}_{2t})\cos\theta_t + Y_t\sin\theta_t], \tag{49}$$

$$g_t\left(\Gamma_t^{\mathrm{R}}\right) = 2\lambda X_t h_t\,\mathrm{e}^{\Gamma_t^{\mathrm{R}}}\cdot\frac{Y_t\sin\theta_t\left[1 + \mathrm{e}^{2\Gamma_t^{\mathrm{R}}}\right] + X_t\cos\theta_t\left[1 - \mathrm{e}^{2\Gamma_t^{\mathrm{R}}}\right]}{\left[1 + \mathrm{e}^{2\Gamma_t^{\mathrm{R}}} + 2h_t\cos\theta_t\,\mathrm{e}^{\Gamma_t^{\mathrm{R}}}\right]\left[1 + \mathrm{e}^{2\Gamma_t^{\mathrm{R}}}\right]} \tag{50}$$

and

$$h_t = \exp\left[-\frac{a_t^{\mathrm{R}}}{2}\left(X_t^2 + Y_t^2\right)\right], \tag{51}$$

$$Y_t = -\frac{2a_t^{\mathrm{I}}X_t + K_t}{2a_t^{\mathrm{R}}}, \tag{52}$$

[10] We will see in section 7 that at the macroscopic level the spread of a Gaussian wavefunction converges very rapidly towards its asymptotic value.

[11] Accordingly, the collapse mechanism is precisely the same as the one of the original GRW model [1].

$$\theta_t = \frac{1}{2}\left(\overline{x}_{1t} + \overline{x}_{2t}\right) K_t + \Gamma_t^{\mathrm{I}}, \qquad \Gamma_t^{\mathrm{I}} = \gamma_{2t}^{\mathrm{I}} - \gamma_{1t}^{\mathrm{I}}, \tag{53}$$

with $a_t^{\mathrm{R}} \equiv a_{1t}^{\mathrm{R}} = a_{2t}^{\mathrm{R}}$.

Equation (48) cannot be solved exactly, due to the presence of the term $g_t\left(\Gamma_t^{\mathrm{R}}\right)$ which is a non-simple function of Γ_t^{R}; to circumvent this problem, we proceed as follows. We study the following stochastic differential equation:

$$\mathrm{d}\tilde{\Gamma}_t^{\mathrm{R}} = \lambda X_t^2 \tanh \tilde{\Gamma}_t^{\mathrm{R}}\,\mathrm{d}t + \sqrt{\lambda} X_t\,\mathrm{d}W_t, \tag{54}$$

which corresponds to equation (48) without the term $g_t\left(\Gamma_t^{\mathrm{R}}\right)$, and at the end of the subsection we estimate the error made by ignoring such a term.

In studying equation (54), it is convenient to introduce the following time change:

$$t \longrightarrow s_t = \lambda \int_0^t X_u^2\,\mathrm{d}u. \tag{55}$$

X_t^2 is a continuous, differentiable and non-negative function, which we can assume not to be identically zero[12] in any sub–interval of $\mathbb{R}^+$; as a consequence, s_t is a monotone increasing—thus invertible—function of t and equation (55) defines a good time change.

Under this time substitution equation (54) becomes

$$\mathrm{d}\tilde{\Gamma}_s^{\mathrm{R}} = \tanh \tilde{\Gamma}_s^{\mathrm{R}}\,\mathrm{d}s + \mathrm{d}\tilde{W}_s, \tag{56}$$

where

$$\tilde{W}_s = \sqrt{\lambda} \int_0^s X_t\,\mathrm{d}W_t \tag{57}$$

is a Wiener process[13] with respect to the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_s : s \geqslant 0\}, \mathbb{P})$. Note that, since according to equation (44) X_t in general decays exponentially in time, $s_\infty < \infty$ and equation (56) is physically meaningful only within the interval[14] $[0, s_\infty)$.

Equations (56) can be analysed in great detail [25]; in particular, the following properties can be proven to hold[15]:

1. Let us define the *collapse time* $S_b \equiv \inf\left\{s : \left|\tilde{\Gamma}_s^{\mathrm{R}}\right| \geqslant b\right\}$: this time is finite with probability 1 and its *average value* is equal to[16]:

$$\mathbb{E}[S_b] = b \tanh b - b_0 \tanh b_0. \tag{58}$$

If $b_0 \ll b$ (which occurs when both terms of the superposition give a non-vanishing contribution), and since we have assumed $b \simeq 10$, then $\mathbb{E}[S_b] \simeq b$.

2. The *variance* $\mathbb{V}[S_b] \equiv \mathbb{E}\left[S_b^2\right] - \mathbb{E}^2[S_b]$ is given by[17]:

$$\mathbb{V}[S_b] = F(b) - F(b_0), \tag{59}$$

with

$$F(x) = x^2 \tanh^2 x + x \tanh x - x^2. \tag{60}$$

Note that since $F(x)$ is an even, positive function, increasing for positive values of x, it follows that $\mathbb{V}[S_b] > 0$ whenever $|b_0| < b$, as it assumed to be.

[12] If there exists an interval I of $\mathbb{R}^+$ such that $X_t \equiv 0\ \forall t \in I$, then equation (39) implies that X_t remains equal to 0 for any subsequent time, i.e., the two Gaussian wavefunctions coincide.

[13] See pp 111–3 of [25].

[14] Thus, strictly speaking, equation (57) defines a Wiener process only for $s \leqslant s_\infty$; we then extend in a standard way the process to the interval (s_∞, ∞).

[15] Throughout the analysis, we will assume that $\tilde{\Gamma}_0^{\mathrm{R}} \equiv b_0 \in (-b, b)$.

[16] See theorem 2, p 108 of [25].

[17] See theorem 3, p 109 of [25].

3. The *collapse probability* P_{ϕ_2} that ϕ^{D} is reduced to ϕ_2, i.e., that $\tilde{\Gamma}_s^{\mathrm{R}}$ hits point b before point $-b$ is given by[18]:

$$P_{\phi_2} = \frac{1}{2}\frac{\tanh b + \tanh b_0}{\tanh b}; \tag{61}$$

according to our choice for b, $\tanh b \simeq 1$ and consequently

$$P_{\phi_2} \simeq \frac{1}{2}[1 + \tanh b_0] = \frac{\mathrm{e}^{2\gamma_{20}^{\mathrm{R}}}}{\mathrm{e}^{2\gamma_{10}^{\mathrm{R}}} + \mathrm{e}^{2\gamma_{20}^{\mathrm{R}}}} = \frac{\|\phi_{20}\|^2}{\|\phi_{10}\|^2 + \|\phi_{20}\|^2}, \tag{62}$$

which (neglecting the overlapping between the two Gaussian wavefunctions) corresponds to the *standard* quantum prescription for the probability that ϕ^{D} collapses to ϕ_2.

4. The *delocalization probability*, i.e., the probability $P_{\phi_2}^{\mathrm{Del}}$ that $\tilde{\Gamma}_t^{\mathrm{R}}$ goes below $b - \eta$ before time s_∞ after having reached b (η is a positive quantity smaller than b), is[19]

$$P_{\phi_2}^{\mathrm{Del}} \leqslant 1 - P\left\{\inf \tilde{\Gamma}_s^{\mathrm{R}} > b - \eta\right\} = 1 - (1 + \tanh b)\frac{\tanh \eta}{1 + \tanh \eta}. \tag{63}$$

In the above definition, we have required a delocalization to occur before time s_∞ because—since s_∞ corresponds to $t = \infty$—a delocalization at a time $\geqslant s_\infty$ does not correspond to a real physical delocalization.

If, for example, we take $\eta = 3$, we have that $P_{\phi_2}^{\mathrm{Del}} \lesssim 0.002$.

This concludes our analysis of the statistical behaviour of $\tilde{\Gamma}_s^{\mathrm{R}}$. We end this section by discussing how one can estimate the error made in neglecting the term $g_t\left(\Gamma_t^{\mathrm{R}}\right)$ in equation (48), i.e. in studying $\tilde{\Gamma}_s^{\mathrm{R}}$ in place of Γ_s^{R}. In the appendix the following estimate for $g_t\left(\Gamma_t^{\mathrm{R}}\right)$ is given

$$\left|g_t\left(\Gamma_t^{\mathrm{R}}\right)\right| \leqslant \lambda \frac{(|X_t| + |Y_t|)^2}{\exp\left(a_t^{\mathrm{R}}(|X_t| + |Y_t|)^2/4\right) - 1} \tag{64}$$

$$\leqslant \frac{\lambda X_t^2}{\exp\left(a_m^{\mathrm{R}} X_m^2/4\right) - 1}, \tag{65}$$

where a_m^{R} and X_m are the minimum values a_t^{R} and X_t take during the time interval one is considering. For example, if we take a 1 g object and we assume that $a_m^{\mathrm{R}} = a_\infty^{\mathrm{R}}$ and $X_m = 1$ cm, then we have:

$$c \equiv \frac{1}{\exp\left(a_m^{\mathrm{R}} X_m^2/4\right) - 1} \simeq \mathrm{e}^{-10^{21}}, \tag{66}$$

which is very close to zero.

By using inequality (65) and lemma 4, p 120 of [25], one can easily show that $\Gamma_s^{\mathrm{R}-} \leqslant \Gamma_s^{\mathrm{R}} \leqslant \Gamma_s^{\mathrm{R}+}$, where $\Gamma_s^{\mathrm{R}\pm}$ are solutions of the following two stochastic differential equations:

$$\mathrm{d}\Gamma_s^{\mathrm{R}\pm} = \left[\tanh \Gamma_s^{\mathrm{R}\pm} \pm c\right] \mathrm{d}s + \mathrm{d}\tilde{W}_s, \tag{67}$$

with an obvious meaning of the signs (as before, we have moved from the variable t to the variable s).

Equation (67) can be analysed along the same lines followed in studying equation (56), getting basically the same results, due to the very small value c takes in most relevant physical situations. This kind of analysis will be done in detail in a future paper: there we will study the

[18] See theorem 4, p 110 of [25].
[19] See formula 6, p 117 of [25].

most important case where a macroscopic superposition can be created, i.e. a measurement-like situation in which a macroscopic object acting like a measuring apparatus interacts with a microscopic system being initially in a superposition of two eigenstates of the operator which is measured; we will show that, throughout the interaction, the wavefunction of the apparatus is—with extremely high probability—always localized in space, and that the measurement has a definite outcome with the correct quantum probabilities.

5. General solution: the asymptotic behaviour

In this section, we analyse the asymptotic behaviour of the general solution of equation (1), showing that—as time increases—any wavefunction collapses towards a stationary Gaussian solution.

5.1. The collapse

We have seen in the previous sections that both the single-Gaussian and double-Gaussian solutions asymptotically converge towards a stationary solution having the form

$$\psi_t(x) = \sqrt[4]{\frac{2a_\infty^{\mathrm{R}}}{\pi}}\exp[-a_\infty(x-\overline{x}_t)^2 + \mathrm{i}\overline{k}_t x], \tag{68}$$

with

$$a_\infty = \lim_{t\to+\infty} a_t = \frac{\lambda}{\omega}(1-\mathrm{i}); \tag{69}$$

it becomes then natural to ask whether such a kind of wavefunction is the asymptotic limit of *any* initial wavefunction. The answer is positive [20] as we shall now see by following the same strategy used in [11, 14, 20] to prove convergence of solutions.

Since a wavefunction of the form (68) is an eigenstate of the operator:

$$A = q + \frac{\mathrm{i}-1}{m\omega}p, \tag{70}$$

the proof consists in showing that the variance[20]:

$$\Delta A_t \equiv \langle\psi_t|[A^\dagger - \langle A^\dagger\rangle][A - \langle A\rangle]|\psi_t\rangle \tag{71}$$

converges to 0 for $t \to +\infty$. The following expression for ΔA_t holds:

$$\Delta A_t = \Delta q_t + \frac{2}{m^2\omega^2}\Delta p_t - \frac{2}{m\omega}\Sigma(q, p) - \frac{\hbar}{m\omega}, \tag{72}$$

with

$$\Sigma(q, p) = \tfrac{1}{2}\left[\langle\psi_t|[q - \langle q\rangle][p - \langle p\rangle]|\psi_t\rangle + \langle\psi_t|[p - \langle p\rangle][q - \langle q\rangle]|\psi_t\rangle\right]. \tag{73}$$

After a rather long calculation, one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\Delta A_t] = -\mathbb{E}\left[\omega\Delta A_t + 2\lambda\left(\Delta q_t - \sigma_q^2(\infty)\right)^2 + 2\lambda\left(\Delta q_t - \frac{2}{m\omega}\Sigma(q, p)\right)^2\right] \leqslant 0. \tag{74}$$

Since, by definition, $\mathbb{E}[\Delta A_t]$ is a non-negative quantity, equation (74) is consistent if and only if the right-hand-side asymptotically vanishes. This, in particular, implies that

$$\Delta A_t \underset{t\to+\infty}{\longrightarrow} 0, \tag{75}$$

except for a subset of Ω of measure zero.

[20] In fact, it is easy to prove that $\Delta A_t = 0$ if and only if $|\psi_t\rangle$ is an eigenstate of the operator A, from which it follows that $\Delta A_t \to 0$ if and only if $|\psi_t\rangle$ converges towards an eigenstate of A. In equation (71), $\langle A\rangle \equiv \langle\psi_t|A\psi_t\rangle$, and similarly for $\langle A^\dagger\rangle$.

5.2. The collapse probability

We now analyse the probability that the wavefunction, as a result of the collapse process, lies within a given region of space[21]. To this purpose, let us consider the following probability measure, defined on the Borel sigma-algebra $\mathcal{B}(\mathbb{R})$:

$$\mu_t(\Delta) \equiv \mathbb{E}_{\mathbb{P}}[\| P_\Delta \psi_t \|^2], \tag{76}$$

where P_Δ is the projection operator associated with the Borel subset Δ of $\mathbb{R}$. Such a measure is identified by the density $p_t(x) \equiv \mathbb{E}_{\mathbb{P}}[|\psi_t(x)|^2]$:

$$\mu_t(\Delta) = \int_\Delta p_t(x)\,\mathrm{d}x, \tag{77}$$

and it can easily be shown that the density $p_t(x)$ corresponds to the diagonal element $\langle x|\rho_t|x\rangle$ of the statistical operator[22] $\rho_t \equiv \mathbb{E}_{\mathbb{P}}\,[|\psi_t\rangle\langle\psi_t|]$, which satisfies the Lindblad-type equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\mathrm{i}}{2m\hbar}[p^2, \rho_t] - \frac{\lambda}{2}\,[q, [q, \rho_t]]\,. \tag{78}$$

The solution of the above equation, expressed in terms of the solution $\langle q_1|\rho_S(t)|q_2\rangle$ of the pure Schrödinger equation ($\lambda = 0$), is[23]

$$\langle q_1|\rho(t)|q_2\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}k\int_{-\infty}^{+\infty}\mathrm{d}y\,\exp(-(\mathrm{i}/\hbar)ky)F[k, q_1 - q_2, t]\langle q_1 + y|\rho_S(t)|q_2 + y\rangle, \tag{79}$$

with

$$F[k, x, t] = \exp\left[-\frac{\lambda}{2}t\left(x^2 - \frac{k}{m}xt + \frac{k^2}{3m^2}t^2\right)\right]. \tag{80}$$

The Hermitian symmetry of $\langle q_1|\rho(t)|q_2\rangle$ follows from the fact that $F[k, x, t] = F[-k, -x, t]$; for $\lambda = 0$, we have $F[k, x, t] = 1$ so that $\langle q_1|\rho(t)|q_2\rangle = \langle q_1|\rho_S(t)|q_2\rangle$ as it must be.

From equation (79) it follows that

$$p_t(x) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}k\int_{-\infty}^{+\infty}\mathrm{d}y\,\exp(-(\mathrm{i}/\hbar)ky)F[k, 0, t]p_t^S(x + y) \tag{81}$$

where $p_t^S(x) = \langle x|\rho_S(t)|x\rangle$ is the standard quantum probability density of finding the particle located at x in a position measurement. One can easily perform at the integration over the k variable:

$$p_t(x) = \sqrt{\frac{\alpha_t}{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}y\,\mathrm{e}^{-\alpha_t y^2}\,p_t^S(x + y), \tag{82}$$

with

$$\alpha_t = \frac{3m_0}{2\hbar^2\lambda_0}\frac{m}{t^3} \simeq 10^{43}\left(\frac{m}{\mathrm{kg}}\right)\left(\frac{\mathrm{s}}{t}\right)^3. \tag{83}$$

For a macroscopic object (let us say $m \geqslant 1$ g) and for very long times (e.g., $t \in [0, 10^8$ s $\simeq 3$ years], a time interval which of course is much longer than the time during which an object can be kept isolated so that the free particle approximation holds true) α_t is a very large number, so large that the exponential function in equation (82) is significantly more narrow

[21] Of course, wavefunctions in general do not have a compact support; accordingly saying that a wavefunction lies within a given region of space amounts to saying that it is almost entirely contained within the region, except for small 'tails' spreading out of that region.

[22] The definition $\rho_t \equiv \mathbb{E}_{\mathbb{P}}\,[|\psi_t\rangle\langle\psi_t|]$ is rather formal; see [16, 17] for a rigorous definition.

[23] Reference [1], appendix C, shows how the solution can be obtained.

than $p_t^S(x)$, for most typical wavefunctions (in section 7.1 we will see that the asymptotic spread of the wavefunction of a 1 g object is about 10^{-13} m). Accordingly, this exponential function acts like a Dirac delta and one has $p_t(x) \simeq p_t^S(x)$ with very high accuracy. This in turn implies that

$$\mu_t(\Delta) \simeq \left\| P_\Delta \psi_t^{\mathrm{Sch}} \right\|^2, \tag{84}$$

in other words, the probability measure $\mu_t(\Delta)$ is very close to the *quantum probability* of finding the particle lying in Δ in a position measurement.

We still have to discuss the physical meaning of the probability measure $\mu_t(\Delta)$ defined by equation (76); such a discussion is relevant only at the *macroscopic* level, since we need only macro-objects to be well localized in space (and with the correct quantum probabilities).

As already anticipated the spread of a wavefunction of a macro-object having the mass of, e.g., 1 g reaches in a very short time a value which is close to the asymptotic spread $\sigma_q(\infty) \simeq 10^{-13}$ m; then, if we take for Δ an interval $[a, b]$ whose length is much greater than $\sigma_q(\infty)$—for example, we can take $b - a = 10^{-7}$ m, which is sufficiently small for all practical purposes—only those wavefunctions whose mean lies around Δ give a non-vanishing contribution to $\mu_t(\Delta)$. As a consequence, with the above choices for Δ (and, of course, waiting a time sufficiently long in order for the reduction to have occurred), the measure $\mu_t(\Delta)$ represents a good probability measure that the wavefunction collapses within Δ.

6. Effect of the reducing terms on the microscopic dynamics

In the previous sections, we have studied some analytical properties of the solutions of the stochastic differential equation (1); we now focus our attention on the dynamics for a microscopic particle.

The time evolution of a (free) quantum particle has three characteristic features:

1. The wavefunction is subject to a localization process which, at the micro-level, is extremely slow, almost negligible. For example, with reference to the situation $\mathcal{A}$ we have defined at the end of section 4.1, equation (45) shows that the distance X_t between the centres of the two Gaussian wavefunctions remains practically unaltered for about 10^5 s; under the approximation $X_t \simeq X_0$, equations (55) and (58) imply that the time necessary for one of the two Gaussians to be suppressed is

$$\mathbb{E}\,[T_b] \simeq 10^6 \left(\frac{\mathrm{m}}{X_0}\right)^2 \mathrm{s} \qquad \text{for an electron,}$$

$$\mathbb{E}\,[T_b] \simeq 10^3 \left(\frac{\mathrm{m}}{X_0}\right)^2 \mathrm{s} \qquad \text{for a nucleon,}$$

which are very long times[24] compared with the characteristic times of a quantum experiment.

2. The spread of the wavefunction reaches an asymptotic value which depends on the mass of the particle:

$$\sigma_q(\infty) \simeq 1\ \mathrm{m} \qquad \text{for an electron,}$$

$$\sigma_q(\infty) \simeq 1\ \mathrm{cm} \qquad \text{for a nucleon.}$$

[24] In the first case, indeed, the reduction time is longer than the time during which the approximation $X_t \simeq X_0$ is valid—see equation (45).

Table 1. λ in $cm^{-2}\ s^{-1}$ for decoherence arising from different kinds of scattering processes (taken from Joos and Zeh [26]). In the last line: λ for the collapse model is given by equation (1).

Cause of decoherence	10^{-3} cm dust particle	10^{-6} cm large molecule
Air molecules	10^{36}	10^{30}
Laboratory vacuum	10^{23}	10^{17}
Sunlight on earth	10^{21}	10^{13}
300 K photons	10^{19}	10^{6}
Cosmic background radiation	10^{6}	10^{-12}
Collapse	10^{7}	10^{-2}

3. The wavefunction undergoes a random motion both in position and momentum under the influence of the stochastic process. These fluctuations can be quite relevant at the microscopic level; for example, for a stationary solution, one has, from equation (33),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{V}[\langle q\rangle_t] \geqslant 1\mathrm{m}^2\ \mathrm{s}^{-1} \qquad \text{for an electron,}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{V}[\langle q\rangle_t] \geqslant 10^{-3}\ \mathrm{m}^2\ \mathrm{s}^{-1} \qquad \text{for a nucleon.}$$

This is the physical picture of a microscopic particle as it emerges from the collapse model. In order for the model to be physically consistent, it must reproduce at the microscopic level the predictions of standard quantum mechanics, an issue which we are now going to discuss.

Within the collapse model measurable quantities are given by averages of the form $\mathbb{E}_{\mathbb{P}}[\langle O\rangle_t]$, where O is (in principle) any self-adjoint operator. It is not difficult to prove that

$$\mathbb{E}_{\mathbb{P}}\,[\langle O\rangle_t] = \mathrm{Tr}\,[O\rho_t]\,, \tag{85}$$

where the statistical operator $\rho_t \equiv \mathbb{E}_{\mathbb{P}}\,[|\psi_t\rangle\langle\psi_t|]$ satisfies the Lindblad-type equation (78). This is a typical master equation used in decoherence theory to describe the interaction between a quantum particle and the surrounding environment [26]; consequently, as far as *experimental results* are concerned, the predictions of our model are similar to those of decoherence models[25]. It becomes then natural to compare the strength of the collapse mechanism (measured by the parameter λ) with that of decoherence.

Such a comparison is given in table 1, when the system under study is a very small particle like an electron or an almost macroscopic object like a dust particle. We see that, for most sources of decoherence, the *experimentally testable* effects of the collapse mechanism are weaker than those produced by the interaction with the surrounding environment. This implies that, in order to test such effects, one has to isolate a quantum system—for a sufficiently long time—from almost all conceivable sources of decoherence, which is quite difficult: the experimentally testable differences between our collapse model and standard quantum mechanics are so small that they cannot be detected unless very sophisticated experiments are performed [22, 23].

[25] We recall the conceptual difference between collapse models and decoherence models. Within collapse models, one modifies quantum mechanics (by adding appropriate nonlinear and stochastic terms) so that macro-objects are always localized in space. Decoherence models, on the other hand, are quantum mechanical models applied to the study of open quantum system; since they assume the validity of the Schrödinger equation, they cannot induce the collapse of the wavefunction of macroscopic systems (as it has been shown, e.g., in [27]) even if one of the effects of the interaction with the environment is to *hide*—not to *eliminate*—macroscopic superpositions in measurement-like situations.

7. Multi-particle systems: effect of the reducing terms on the macroscopic dynamics

The generalization of equation (1) to a system of N interacting and distinguishable particles is straightforward:

$$\mathrm{d}\psi_t(\{x\}) = \left[-\frac{\mathrm{i}}{\hbar}H_{\mathrm{T}}\,\mathrm{d}t + \sum_{n=1}^{N}\sqrt{\lambda_n}(q_n - \langle q_n\rangle_t)\,\mathrm{d}W_t^n - \frac{1}{2}\sum_{n=1}^{N}\lambda_n(q_n - \langle q_n\rangle_t)^2\,\mathrm{d}t\right]\psi_t(\{x\}), \tag{86}$$

where H_{T} is the standard quantum Hamiltonian of the composite system, the operators q_n $(n = 1, \ldots, N)$ are the position operators of the particles of the system and W_t^n $(n = 1, \ldots, N)$ are N independent standard Wiener processes; the symbol $\{x\}$ denotes the N spatial coordinates $x_1, \ldots, x_N$.

For the purposes of our analysis, it is convenient to switch to the centre-of-mass (R) and relative $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)$ coordinates:

$$\begin{cases} R = \dfrac{1}{M}\displaystyle\sum_{n=1}^{N} m_n x_n \qquad M = \sum_{n=1}^{N} m_n, \\ x_n = R + \tilde{x}_n; \end{cases} \tag{87}$$

let Q be the position operator for the centre of mass and $\tilde{q}_n$ $(n = 1, \ldots, N)$ the position operators associated with the relative coordinates.

It is not difficult to show that—under the assumption $H_{\mathrm{T}} = H_{\mathrm{CM}} + H_{\mathrm{rel}}$—the dynamics for the centre of mass and that for the relative motion decouple; in other words, $\psi_t(\{x\}) = \psi_t^{\mathrm{CM}}(R)\otimes\psi_t^{\mathrm{rel}}(\{\tilde{x}\})$ solves equation (86) whenever $\psi_t^{\mathrm{CM}}(R)$ and $\psi_t^{\mathrm{rel}}(\{\tilde{x}\})$ satisfy the following equations:

$$\mathrm{d}\psi_t^{\mathrm{rel}}(\{\tilde{x}\}) = \left[-\frac{\mathrm{i}}{\hbar}H_{\mathrm{rel}}\,\mathrm{d}t + \sum_{n=1}^{N}\sqrt{\lambda_n}(\tilde{q}_n - \langle\tilde{q}_n\rangle_t)\,\mathrm{d}W_t^n - \frac{1}{2}\sum_{n=1}^{N}\lambda_n(\tilde{q}_n - \langle\tilde{q}_n\rangle_t)^2\,\mathrm{d}t\right]\psi_t^{\mathrm{rel}}(\{x\}), \tag{88}$$

$$\mathrm{d}\psi_t^{\mathrm{CM}}(R) = \left[-\frac{\mathrm{i}}{\hbar}H_{\mathrm{CM}}\,\mathrm{d}t + \sqrt{\lambda_{\mathrm{CM}}}(Q - \langle Q\rangle_t)\,\mathrm{d}W_t - \frac{\lambda_{\mathrm{CM}}}{2}(Q - \langle Q\rangle_t)^2\,\mathrm{d}t\right]\psi_t^{\mathrm{CM}}(R), \tag{89}$$

with

$$\lambda_{\mathrm{CM}} = \sum_{n=1}^{N}\lambda_n = \frac{M}{m_0}\lambda_0. \tag{90}$$

The first of the above equations describes the internal motion of the composite system, and will not be analysed in this paper; in the remainder of the section we will focus our attention on the second equation.

Equation (89) shows that the reducing terms associated with the centre of mass of a composite system are equal to those associated with a particle having mass equal to the total mass M of the composite system; in particular, when the system is isolated—i.e., $H_{\mathrm{CM}} = P^2/2M$, where P is the total momentum—the centre of mass behaves like a free particle whose dynamics has been already analysed in secctions 3–5. In the next subsections, we will see that, because of the large mass of a macro-object, the dynamics of its centre of mass is radically different from that of a microscopic particle.

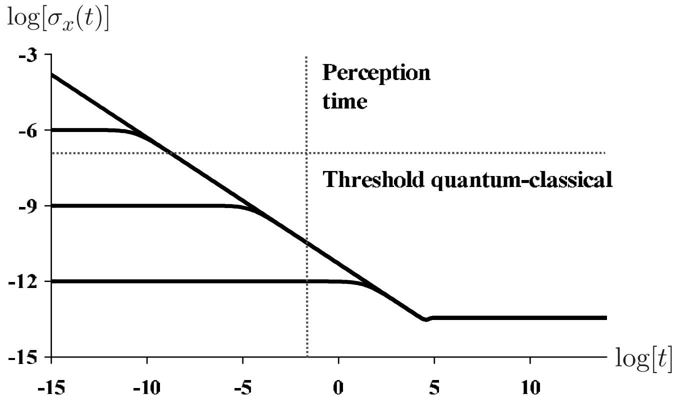

Figure 2. Time evolution of the spread $\sigma_q(t)$ of a Gaussian wavefunction of the centre of mass of a system containing $N = 10^{24}$ nucleons; different initial conditions have been considered. Time is measured in seconds, while the spread is measured in meters.

7.1. The amplification mechanism

The first important feature of collapse models is what has been called the amplification mechanism [1, 2]: the reduction rates of the constituents of a macro-object sum up, so that the reduction rate associated with its centre of mass is much greater than the reduction rates of the single constituents.

This situation is exemplified in figure 2, which shows the time evolution of the spread $\sigma_q(t)$ of a Gaussian wavefunction. We note that, however large the initial wavefunction is, after less than 10^{-2} s—which corresponds to the perception time of a human being—its spread goes below 10^{-5} cm, which is the threshold chosen in the original GRW model [1] below which a wavefunction can be considered sufficiently well localized to describe the classical behaviour of a macroscopic system.

More generally equation (74) implies that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\Delta A_t\right]\right| \geqslant 2\lambda\left[\Delta q - \sigma_q^2(\infty)\right]^2, \tag{91}$$

and, as long as the spread of the wavefunction is significantly greater than its asymptotic value, i.e., the wavefunction is not already sufficiently well localized in space, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\Delta A_t]\right| \geqslant 10^{25}\left(\frac{m}{\mathrm{kg}}\right)\left(\frac{\Delta q}{\mathrm{m}}\right)^2 \mathrm{m}^2\,\mathrm{s}^{-1}, \tag{92}$$

which is a very high reduction rate, for a macroscopic object. Note that, as previously stated, the velocity increases, for increasing values of the mass of the particle.

Asymptotically, the wavefunction of a macro-object has an extremely small spread; for example,

$$\sigma_q(\infty) \simeq 10^{-13}\,\mathrm{m} \qquad \text{for} \quad \text{a 1 g object,}$$
$$\sigma_q(\infty) \simeq 10^{-27}\,\mathrm{m} \qquad \text{for} \quad \text{the Earth.}$$

Thus, according to our collapse model, macro-particles behave like point-like particles.

7.2. Damping of fluctuations

We have seen that the mean (both in position and in momentum) of the wavefunction undergoes a diffusion process arising from the stochastic dynamics: such a diffusion is quite important at the microscopic level, and it is responsible for the agreement of the physical predictions of the model with those given by standard quantum mechanics. We now analyse the magnitude of the fluctuations at the macroscopic level.

Contrary to the behaviour of the reduction mechanism, which is amplified when moving from the micro- to the macro-level, the fluctuations associated with the motion of microscopic particles interfere destructively with each other, in such a way that the diffusion process associated with the centre of mass of an N-particle system is much weaker than those of the single components. We now give some estimates.

Let us suppose that the centre-of-mass wavefunction has reached a stationary solution; under this assumption one has from equations (33)–(35):

$$\mathbb{V}[\langle q\rangle_t] = \frac{\omega}{8\lambda}\left[\frac{(\omega t)^3}{3!} + \frac{(\omega t)^2}{2!} + \omega t\right], \tag{93}$$

$$\mathbb{V}[\langle p\rangle_t] = \lambda\hbar^2 t. \tag{94}$$

Since, for example, $\omega/8\lambda \simeq 10^{-27}$ m^2 for a 1 g object, and $\omega/8\lambda \simeq 10^{-54}$ m^2 for the Earth, we see that for a macro-object the numerical values of the parameters are such that for very long times (in many cases much longer that the age of the universe) the fluctuations are so small that, for all practical purposes, they can be neglected; this is how classical determinism is recovered within our stochastic model.

The above results imply that the actual values of $\langle q\rangle_t$ and $\langle p\rangle_t$ are practically equivalent to their stochastic averages; since these stochastic averages obey the classical laws of motion (31) and (32), we find out that $\langle q\rangle_t$ and $\langle p\rangle_t$ practically evolve according to the classical laws of motion, for most realizations of the stochastic process.

The conclusion is the following: in the macroscopic regime the wavefunction of a macroscopic system behaves, for all practical purposes, like a point-like particle moving deterministically according to Newton's laws of motion.

8. Conclusions

From the analysis of the previous sections we have seen that, in general, the evolution of the wavefunction as predicted by the collapse model is significantly different from that predicted by standard quantum mechanics, both at the micro- and at the macro-level. For example, at the microscopic level the random fluctuations can be very large, while in the standard case there are no fluctuations; at the macroscopic level, wavefunctions rapidly localize in space, while in the standard quantum case they keep spreading.

Anyway, as far as *physical predictions* are concerned, our model is almost equivalent to standard quantum mechanics, the differences being so small that they can hardly be detected with present-day technology. Moreover, at the macroscopic level the localization mechanism becomes very rapid and the fluctuations almost disappear: the wavefunction of the centre of mass of a macroscopic object behaves like a point-like particle moving according to Newton's laws.

To conclude, the stochastic model reproduces, with excellent accuracy, both quantum mechanics at the microscopic level and classical mechanics at the macroscopic one, and also describes the transition from the quantum to the classical domain.

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Appendix. Derivation of inequality (64)

From equation (49) one has

$$\left|g_t\left(\Gamma_t^{\mathrm{R}}\right)\right| \leqslant 2\lambda|X_t|h_t\left[|Y_t\sin\theta_t|\left|\frac{\mathrm{e}^{\Gamma_t^{\mathrm{R}}}}{1+\mathrm{e}^{2\Gamma_t^{\mathrm{R}}}+2h_t\cos\theta_t\,\mathrm{e}^{\Gamma_t^{\mathrm{R}}}}\right|\right.$$
$$\left.+|X_t\cos\theta_t|\left|\frac{\mathrm{e}^{\Gamma_t^{\mathrm{R}}}}{1+\mathrm{e}^{2\Gamma_t^{\mathrm{R}}}+2h_t\cos\theta_t\,\mathrm{e}^{\Gamma_t^{\mathrm{R}}}}\right|\left|\frac{1-\mathrm{e}^{2\Gamma_t^{\mathrm{R}}}}{1+\mathrm{e}^{2\Gamma_t^{\mathrm{R}}}}\right|\right]. \tag{A.1}$$

By using the fact that $\sin\theta_t \leqslant 1$ and $\cos\theta_t \leqslant 1$, that the function

$$f_1(x) = \frac{x}{1+x^2+2cx} \qquad x \geqslant 0, \quad -1 < c \leqslant 1 \tag{A.2}$$

is bounded between 0 and $1/(2+2c)$, while the function

$$f_2(x) = \frac{1-x}{1+x} \qquad x \geqslant 0 \tag{A.3}$$

is bounded between 1 and -1, we get

$$\left|g_t\left(\Gamma_t^{\mathrm{R}}\right)\right| \leqslant \lambda\frac{h_t}{1-h_t}\left[X_t^2+|X_tY_t|\right] \leqslant \lambda\frac{h_t}{1-h_t}\left[|X_t|+|Y_t|\right]^2, \tag{A.4}$$

from which, dividing both the numerator and the denumerator on the right-hand-side by h_t and using $(x+y)^2 \leqslant 2(x^2+y^2)$, inequality (64) follows.

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